Discipline: **Physics**
Subject: **Electromagnetic Theory**
Unit 31:
Lesson/ Module: **Multipole Fields - II**

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Learning Objectives:

From this module, a continuation of the last module, students may get to know about the following:

1. *Properties of multipole radiation.*
2. *Energy contained in the electric and magnetic multipole radiation.*
3. *The angular momentum of the electric and magnetic multipole radiation.*
4. *Detailed angular distribution of the electric and magnetic multipole radiation.*

31 Multipole Fields - II

31.1 Electric and magnetic multipole fields

In the last unit we had started with a general discussion of multipole fields. We had obtained the spherical wave solution of the scalar wave equation in terms of the spherical Bessel functions. We had then introduced the spherical harmonics and the operator $\vec{L} = -i(\vec{r} \times \vec{\nabla})$ which is familiar to the students from quantum mechanics. Next we described some properties of this operator. Further we took up the task of finding the spherical wave solution of Maxwell equations. These being vector equations, their solutions were also obtained in terms of the vector spherical harmonics $\vec{L}Y_l^m$. We have two types of solutions- electric multipole and magnetic multipole solutions:

$$\vec{E}_{l,m}^{(M)}(r,\theta,\phi) = Z_0 g_l(kr)\vec{L}Y_l^m(\theta,\phi) \tag{1}$$

$$\vec{H}_{l,m}^{(M)} = -i\frac{1}{Z_0 k}\vec{\nabla} \times \vec{E}_{l.m}^{(M)} \tag{2}$$

$$\vec{H}_{l,m}^{(E)}(r,\theta,\phi) = f_l(kr)\vec{L}Y_l^m(\theta,\phi) \tag{3}$$

$$\vec{E}_{l,m}^{(E)} = i\frac{Z_0}{k}\vec{\nabla} \times \vec{H}_{l,m}^{(E)} \tag{4}$$

$$Z_0 = \sqrt{\mu_0/\varepsilon_0} = \mu_0 c$$

The normalized vector spherical harmonics are

$$\vec{X}_l^{\,m}(\theta,\phi) = \frac{1}{\sqrt{l(l+1)}}\vec{L}Y_l^m(\theta,\phi), l \neq 0 \tag{5}$$

In terms of these the general solution of the Maxwell equation was written as

$$\vec{H} = \sum_{l,m}[a_E(l,m) f_l(kr)\vec{X}_l^{\,m} - \frac{i}{k}a_M(l,m)\vec{\nabla} \times (g_l(kr)\vec{X}_l^{\,m})] \tag{6}$$

$$\vec{E} = Z_0\sum_{l,m}[a_M(l,m)g_l(kr)\vec{X}_l^{\,m} + \frac{i}{k}a_E(l,m)\vec{\nabla} \times (f_l(kr)\vec{X}_l^{\,m})] \tag{7}$$

with

$$a_M(l,m)g_l(kr) = \frac{k}{\sqrt{l(l+1)}}\int Y_l^{m*}(\vec{r}.\vec{H})d\Omega \tag{8}$$

$$a_E(l,m)\,f_l(kr) = -\frac{k}{Z_0\sqrt{l(l+1)}}\int Y_l^{m*}(\vec{r}.\vec{E})d\Omega \qquad (9)$$

We now continue with the study of multipole fields. Our next task is to find the properties of these multipole fields, viz. energy and angular momentum etc.

31.2 Properties of multipole fields

Let us look at the individual multipole fields given by equations (1) to (4). In equation (3) for example, $f_l(kr)$ is a linear combination of the two spherical Bessel functions n_l and j_l. In the near zone $kr << 1$, the function j_l tends to zero and from the limiting behaviour of n_l,

$$n_l(x) \to -\frac{(2l-1)!!}{x^{l+1}} \qquad (10)$$

we have

$$\vec{H}_{l,m}^{(E)} \to -\frac{k}{l}\vec{L}\frac{Y_l^m}{r^{l+1}} \qquad (11)$$

where the proportionality coefficient $-\frac{k}{l}$ is chosen for later convenience. The electric field, given by equation (4), becomes

$$\vec{E}_{l,m}^{(E)} \to -\frac{iZ_0}{l}\vec{\nabla}\times\vec{L}\frac{Y_l^m}{r^{l+1}} \qquad (12)$$

To find the curl on the right hand side we use the vector identity

$$i\vec{\nabla}\times\vec{L} = \vec{\nabla}\times(\vec{r}\times\vec{\nabla}) = \vec{r}\nabla^2 - \vec{\nabla}(1+r\frac{\partial}{\partial r}) \qquad (13)$$

Since $\frac{Y_l^m}{r^{l+1}}$ is a solution of the Laplace equation, the first term vanishes. The result is

$$\vec{E}_{l,m}^{(E)} \to -Z_0\vec{\nabla}(\frac{Y_l^m}{r^{l+1}}) \qquad (14)$$

The presence of gradient gives one higher power of r in the denominator so that in the near zone the electric field dominates over the magnetic field $\vec{H}$. The magnetic multipole fields can be obtained in exactly the same manner or by simply noting that the roles of $\vec{E}$ and $\vec{H}$ are interchanged according to the transformation

$$\vec{E}^{(E)} \rightarrow -Z_0 \vec{H}^{(M)}, \qquad \vec{H}^{(E)} \rightarrow \vec{E}^{(M)} / Z_0 \tag{15}$$

In the far or radiation zone (*kr*>>1) the multipole fields depend on the boundary conditions imposed. Since we are interested in radiation by sources, we choose to consider outgoing wave boundary condition. Then the radial wave function $f_l(kr)$ is proportional to $h_l^{(1)}(kr)$. On using the asymptotic form of $h_l^{(1)}(kr)$:

$$h_l^{(1)}(x) \rightarrow (-i)^{l+1} \frac{e^{ix}}{x} \tag{16}$$

we see that in the far zone the magnetic field for an electric (*l*, *m*) multipole goes as

$$\vec{H}_{l,m}^{(E)} \rightarrow (-i)^{(l+1)} \frac{e^{ikr}}{kr} \vec{L} Y_l^m \tag{17}$$

The electric field is given by

$$E_{l,m}^{(E)} \rightarrow \frac{Z_0 (-i)^l}{k^2} [\vec{\nabla}(\frac{e^{ikr}}{r}) \times \vec{L} Y_l^m + \frac{e^{ikr}}{r} \vec{\nabla} \times \vec{L} Y_l^m] \tag{18}$$

Since we are keeping only the highest power of *r* in our asymptotic formula,

$$\vec{\nabla}(\frac{e^{ikr}}{r}) = ik \frac{e^{ikr}}{r} \hat{n}$$

In the second term use the vector identity for $\vec{\nabla} \times \vec{L}$, equation (13). The result is

$$\vec{E}_{l,m}^{(E)} = -(-i)^{l+1} \frac{Z_0 e^{ikr}}{kr} [\hat{n} \times \vec{L} Y_l^m - \frac{1}{k} (\vec{r} \nabla^2 - \vec{\nabla}) Y_l^m] \tag{19}$$

Since $\nabla \sim 1/r$, the second term can be omitted in the limit *kr*>>1, and we get

$$\vec{E}_{l,m}^{(E)} = Z_0 \vec{H}_{l,m}^{(E)} \times \hat{n} \tag{20}$$

These fields are typical radiation fields, falling off as 1/*r*.

For the magnetic multipoles, the corresponding results are obtained by making the interchange (15).

31.2.1 Energy in the multipole radiation

The fields that we have found above can now be used to calculate the energy and angular momentum of the radiation. Let us consider a linear superposition of electric multipole fields with definite *l* but different *m* values. Use the expression (6) above for the magnetic field

$$\vec{H} = \sum_{l,m} [a_E(l,m) f_l(kr) \vec{X}_l^{\,m} - \frac{i}{k} a_M(l,m) \vec{\nabla} \times (g_l(kr) \vec{X}_l^{\,m})], \qquad (21)$$

$$\vec{X}_l^{\,m}(\theta,\phi) = \frac{1}{\sqrt{l(l+1)}} \vec{L} Y_l^m(\theta,\phi), l \neq 0; \qquad (22)$$

$$\int \vec{X}_{l'}^{\,m'*} . \vec{X}_l^{\,m} d\Omega = \delta_{ll'} \delta_{mm'} \qquad (23)$$

replace $f_l(kr)$ by $h_l^{(1)}(kr)$, and we have

$$\vec{H}_l = \sum_m a_E(l,m) \vec{X}_l^m h_l^{(1)}(kr) e^{-i\omega t}$$
$$\vec{E}_l = \frac{iZ_0}{k} \vec{\nabla} \times \vec{H}_l \qquad (24)$$

For harmonically varying fields the time-averaged energy density has been derived earlier, and is

$$< u >= \frac{1}{4} \varepsilon_0 (\vec{E}.\vec{E}* + Z_0{}^2 \vec{H}.\vec{H}*) = \frac{1}{2} \varepsilon_0 Z_0{}^2 \vec{H}.\vec{H}* = \frac{1}{2} \mu_0 \vec{H}.\vec{H}* \qquad (25)$$

The last form follows from the fact that in the radiation zone the two terms are equal. Now replacing $h_l^{(1)}(kr)$ by its asymptotic form, equation (16), the energy in a spherical shell between *r* and (*r* + *dr*) is

$$dU = r^2 dr \int < u > d\Omega = r^2 dr \int \frac{1}{2} \mu_0 \vec{H}.\vec{H}* d\Omega$$
$$= \frac{\mu_0 dr}{2k^2} \sum_{m,m'} a_E*(l,m') a_E(l,m) \int \vec{X}_l^{m'*} . \vec{X}_l^m d\Omega \qquad (26)$$

Further using the normalization equation (23), we get

$$\frac{dU}{dr} = \frac{\mu_0}{2k^2} \sum_m |a_E(l,m)|^2 \qquad (27)$$

independent of radius as was to be expected.

For a general superposition of electric and magnetic monopoles of all *l*-values, there will be summation over *l* as well, and corresponding magnetic multipoles will also be added:

$$\frac{dU}{dr} = \frac{\mu_0}{2k^2} \sum_{l,m} [|a_E(l,m)|^2 + |a_M(l,m)|^2] \qquad (28)$$

Thus the total energy in a spherical shell in the radiation zone is an *incoherent sum* over all multipoles.

31.2.2 Angular momentum of the multipole radiation

Let us now find the angular momentum carried by the radiation. The density of angular momentum of the field is (we use the symbol $\vec{m}$ for density of angular momentum of the field, and $\vec{M}$ for the total angular momentum, to avoid confusion with the angular momentum operator $\vec{L}$.)

$$\vec{m} = \vec{r} \times \vec{p}_{field} = \varepsilon_0 \vec{r} \times (\vec{E} \times \vec{B}) = \varepsilon_0 \mu_0 \vec{r} \times (\vec{E} \times \vec{H}) = \frac{1}{c^2} \vec{r} \times (\vec{E} \times \vec{H}) \tag{29}$$

For harmonic fields the time-averaged angular momentum density will be

$$< \vec{m} > = \frac{1}{2c^2} \mathrm{Re}[\vec{r} \times (\vec{E} \times \vec{H}^*)] \tag{30}$$

Expand the triple vector product and use equation (24) for the electric field to get

$$< \vec{m} > = \frac{\mu_0}{2\omega} \mathrm{Re}[\vec{H}^* (\vec{L}.\vec{H})] \tag{31}$$

Once again let us first consider a linear superposition of electric multipole fields with definite l but different m values.The total field angular momentum in a spherical shell between r and (r + dr) in the far zone is

$$\begin{aligned} d\vec{M} &= r^2 dr \int < \vec{m} > d\Omega = r^2 dr \int \frac{\mu_0}{2\omega} \mathrm{Re}[\vec{H}^* (\vec{L}.\vec{H})] d\Omega \\ &= \frac{dr \mu_0}{2\omega k^2} \mathrm{Re} \sum_{m,m'} a_E^*(l,m') a_E(l,m) \int (\vec{L}.\vec{X}_l^{m'})^* \vec{X}_l^m d\Omega \end{aligned} \tag{32}$$

Now use the explicit form, equation (22), for $\vec{X}_l^m$ and rewrite the above equation as

$$\frac{d\vec{M}}{dr} = \frac{\mu_0}{2\omega k^2} \mathrm{Re} \sum_{m,m'} a_E^*(l,m') a_E(l,m) \int Y_l^{m'*} \vec{L} Y_l^m d\Omega \tag{33}$$

On using the orthonormality of spherical harmonics and the properties of $\vec{L}Y_l^m$ studied in the last module:

$$\int Y_{l'}^{m'*} Y_l^m d\Omega = \delta_{l'l} \delta_{m'm} \tag{34}$$

$$\begin{aligned} L_\pm Y_l^m &= \sqrt{(l \mp m)(l \pm m + 1)} Y_l^{m\pm1} \\ L_z Y_l^m &= m Y_l^m \end{aligned} , \tag{35}$$

we can obtain the following expressions for the components of $\frac{d\vec{L}}{dr}$:

$$\frac{dM_x}{dr} = \frac{\mu_0}{4\omega k^2} \mathrm{Re} \sum_m [\sqrt{(l-m)(l+m+1)} a_E^*(l,m+1) + \sqrt{(l+m)(l-m+1)} a_E^*(l,m-1)] a_E(l,m) \quad (36)$$

$$\frac{dM_y}{dr} = \frac{\mu_0}{4\omega k^2} \mathrm{Im} \sum_m [\sqrt{(l-m)(l+m+1)} a_E^*(l,m+1) - \sqrt{(l+m)(l-m+1)} a_E^*(l,m-1)] a_E(l,m) \quad (37)$$

$$\frac{dM_z}{dr} = \frac{\mu_0}{2\omega k^2} \mathrm{Re} \sum_m m \,|\, a_E(l,m) \,|^2 \quad (38)$$

The expressions for the *x*- and *y*- components are rather involved, that for the *z*- component is simple. This has to do with the way the spherical polar coordinates are defined where z-axis is taken as the polar axis.

For a multipole with a single *m* value, M_x and M_y both vanish while

$$\frac{dM_z}{dr} = \frac{m}{\omega} \frac{dU}{dr} \quad (39)$$

independent of *r*.

This is a relation between the energy and angular momentum in classical physics. In quantum mechanics these equations have a familiar interpretation. The radiation from a multipole of order (*l*,*m*) carries off $m\hbar$ units of *z* component of angular momentum per photon of energy $\hbar\omega$. Even with a superposition of radiation with different *m*, the same interpretation holds. Now each multipole of definite *m* contributes incoherently to the *z* component of angular momentum. In this case, however, the *x* and *y* components of angular momentum are also nonvanishing, with multipole of adjacent *m* components contributing to the sum. All this is course familiar in quantum mechanics.

Though we have presented the results for the specific case of electric multipole radiation, the same holds for magnetic multipole radiation as well. In fact the result holds in general. If we have a superposition of both electric and magnetic multipoles, equation (32) will be generalized to

$$d\vec{M} = \frac{dr\mu_0}{2\omega k^2} \mathrm{Re} \sum_{m,m'} \{ [a_E^*(l',m') a_E(l,m) + a_M^*(l',m') a_M(l,m)] \int (\vec{L}.\vec{X}_{l'}^{m'})^* \vec{X}_l^m d\Omega + [a_E^*(l',m') a_M(l,m) - a_M^*(l',m') a_E(l,m)] \int (\vec{L}.\vec{X}_{l'}^{m'})^* \hat{n} \times \vec{X}_l^m d\Omega \} \quad (40)$$

When both electric and magnetic multipoles are present, there will be electric and magnetic multipole terms (first line) and the interference term (second line). It can be shown that interference comes from those terms in which the value of l for electric and magnetic multipoles differs by unity.

We continue with our analogy to quantum mechanics. In this regard we would expect the ratio of the square of angular momentum to square of energy to have the value

$$\frac{M^{(q)^2}}{U^2} = \frac{(M_x{}^2 + M_y{}^2 + M_z{}^2)}{U^2} = \frac{l(l+1)}{\omega^2} \qquad (41)$$

On the other hand, as we have seen, the classical result for a pure (l, m) multipole is

$$\frac{M^{(c)^2}}{U^2} = \frac{(M_z{}^2)}{U^2} = \frac{m^2}{\omega^2} \qquad (42)$$

since (M_x, M_y) do no contribute.

The reason for this anomaly lies in quantum nature of the electromagnetic field and the uncertainty principle. According to the uncertainty principle, if the z component of angular momentum is known precisely, the other components will be completely uncertain, because the three components of angular momentum do no commute with each other. However their mean square value is such that equation (41) holds. All this is true in quantum mechanics for a single photon state. If, however, the radiation contains many photons (the classical limit), the mean square value of the transverse components can be made negligible compared to the z component. It can be shown that for a state containing N photons

$$\frac{[M^{(q)}(N)]^2}{[U(N)]^2} = \frac{N^2m^2 + Nl(l+1) - m^2}{N^2\omega^2} \qquad (43)$$

Selection Rules

The quantum-mechanical interpretation of the radiated angular momentum leads to selection rules for transitions between various quantum states. Between two states (J, M) [angular momentum J and z component M] and (J', M'), possible multipole transitions have (l, m) such that $|J - J'| \leq l \leq J + J'$ and $m = M - M'$.

Another aspect regarding the quantum radiative transitions is the parity of a state. Parity is a property which determines the behavior of a field or state under space inversion, $\vec{r} \to -\vec{r}$, and in quantum mechanics is a quantum number that is conserved in electromagnetic transitions. The parity of a multipole field is determined by the magnetic field $\vec{H}_{l,m}$. The interaction of a charged particle with the electromagnetic field is proportional to $(\vec{v}.\vec{A})$. Since $\vec{H} = \vec{\nabla} \times \vec{A}/\mu_0$ and the curl operator changes parity, $\vec{H}_{l,m}$ and $\vec{A}_{l,m}$ have opposite parities. Further since $\vec{v}$ is a polar vector, the operator $\vec{v}.\vec{A}$ will differ in parity by the parity of the magnetic field.

For *electric multipole*, the magnetic field is given by equation (24). Parity transformation, $\vec{r} \to -\vec{r}$, in spherical polar coordinates amounts to $r \to r, \theta \to \pi - \theta, \phi \to \phi + \pi$. The operator $\vec{L} = \vec{r} \times \vec{p}$, being an axial vector, is invariant under inversion, and from its definition, the parity of Y_l^m is $(-1)^l$. Thus the parity of $\vec{H}_{l,m}$ in this case is $(-1)^l$. The electric field, $\vec{E}_{l,m} = \frac{iZ_0}{k} \vec{\nabla} \times \vec{H}_{l,m}$ therefore has parity $(-1)^{l+1}$. Similarly one can see that for a magnetic multipole of order (l,m) the parity is $(-1)^{l+1}$.

Adding the requirement of parity conservation, it follows that only certain transitions are allowed, the rest are forbidden. For example, if the states have $J = 1/2$ and $J' = 3/2$, the allowed multipole orders are $l = 1,2$. If the parities of the two states are same only magnetic dipole and electric quadrupole transitions are allowed. In the event of the two parities being opposite, the allowed transitions are electric dipole and magnetic quadrupole.

31.3 Angular distribution of the multipole radiation

For a general localized source distribution, the radiation will consist of a linear combination of all multipoles, both electric and magnetic. The fields are given by equations (6) and (7) and in the radiation zone these expressions reduce to

$$\vec{B} \to \frac{e^{i(kr-\omega t)}}{kr} \sum_{l,m} (-i)^{l+1} [a_E(l,m) \vec{X}_l^{\,m} + a_M(l,m) \hat{n} \times \vec{X}_l^{\,m}] \qquad (44)$$

$$\vec{E} \to Z_0 \vec{H} \times \hat{n} \qquad (45)$$

The coefficients $a_E(l,m)$ and $a_M(l,m)$ are of course related to the properties of the source. From the expression for the magnetic field we obtain the time-averaged power in the usual way:

$$\frac{dP}{d\Omega} = \frac{\mu_0 c}{2k^2} \left| \sum_{l,m} (-i)^{l+1} [a_E(l,m) \vec{X}_l^{\,m} \times \hat{n} + a_M(l,m) \vec{X}_l^{\,m}] \right|^2 \qquad (46)$$

It follows from this expression that

- The electric and magnetic monopoles of a given order (l,m) have the same angular distribution.
- The polarization states of the magnetic and electric multipoles of a given order (l,m) are at right angles to each other.
- The order of radiation can be determined by measuring the angular distribution.
- The magnet or electric nature of the multipole radiation needs measurement of the polarization as well.

For a pure multipole of order (l, m)

$$\frac{dP(l,m)}{d\Omega} = \frac{\mu_0 c}{2k^2} | a_E(l,m)\vec{X}_l^m \times \hat{n} + a_M(l,m)\vec{X}_l^m |^2 \tag{47}$$

Or

$$\frac{dP(l,m)}{d\Omega} = \frac{\mu_0 c}{2k^2} | a(l,m) |^2 | \vec{X}_l^m |^2; \tag{48}$$
$$| a(l,m) |^2 = | a_E(l,m) |^2 + | a_M(l.m) |^2$$

From the definition (5) of $\vec{X}_l^m$ in terms of Y_l^m and the well known properties of Y_l^m and the operator $\vec{L}$ studied in the last module, we have

$$\frac{dP(l,m)}{d\Omega} = \frac{\mu_0 c}{2k^2} \frac{| a(l,m) |^2}{l(l+1)} \left[\begin{array}{l} \frac{1}{2}(l-m)(l+m+1) | Y_l^{m+1} |^2 \\ + \frac{1}{2}(l+m)(l-m+1) | Y_l^{m-1} |^2 + m^2 | Y_l^m |^2 \end{array} \right] \tag{49}$$

The table below lists the value of the expression in the square bracket for *l*=1, 2:

$l \downarrow$ $m \rightarrow$	0	± 1	± 2
1 (dipole)	$\frac{3}{8\pi}\sin^2\theta$	$\frac{3}{16\pi}(1+\cos^2\theta)$	------
2 (quadrupole)	$\frac{15}{8\pi}\sin^2\theta\cos^2\theta$	$\frac{5}{16\pi}(1-3\cos^2\theta+4\cos^4\theta)$	$\frac{5}{16\pi}(1-\cos^4\theta)$

These distributions are shown in the figure below. **[Figure 16.1 from Jackson ed. 2. Rotate all the five figures by 90^0. The middle figure will remain unchanged.]**

The dipole distribution correspond to an oscillating dipole; parallel to the *z*-axis for m=0; and along the *x*- or *y*-axis for $m = \pm 1$.

Using the addition theorem for spherical harmonics,

$$\sum_{m=-l}^{l} | Y_l^m(\theta,\phi) |^2 = \frac{2l+1}{4\pi}$$

we can show that

$$\sum_{m=-l}^{i} | \vec{X}_l^m(\theta,\phi)|^2 = \frac{2l+1}{4\pi} \tag{50}$$

Thus if a source consists of a set of multipoles of order l with coefficients $a(l,m)$ independent of m, the angular distribution will be isotropic. This is usually the case with atomic and nuclear radiative transitions.

The total power radiated by a pure multipole of order (l,m) is obtained by integrating equation (48) over the entire solid angle and is

$$P(l,m) = \frac{\mu_0 c}{2k^2} | a(l,m)|^2 \tag{51}$$

For a general source the angular distribution is given by equation (46). On integration over the solid angle, the cross term disappears, and the total power radiated is just the incoherent sum of the individual contributions:

$$P = \frac{\mu_0 c}{2k^2} \sum_{l,m} [| a_E(l,m)|^2 + | a_M(l,m)|^2] \tag{52}$$

Summary

1. *In this module we have continued with our study of the multipole radiation.*
2. *Our next task was to study the various properties of the multipole radiation.*
3. *We obtain expressions for the energy density and energy flow in both electric and magnetic multipole radiation.*
4. *Next we study angular momentum in multipole radiation. We work out in detail the case of electric multipole; magnetic multipole case can be dealt with in a similar fashion.*
5. *We also obtain expression for the ratio of angular momentum to energy of the field and comment on the difference between the classical and quantum mechanical expressions for this.*
6. *Finally we look at the detailed angular distribution of the electric and magnetic multipole radiation. We make several comments on the similarities and differences between the angular distribution of electric and magnetic multipoles.*
7. *We tabulate the angular distribution for the two important cases of dipole and quadrupole radiation.*